

RAMANUJAN SERIES UPSIDE-DOWN

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ABSTRACT. We prove that there is a correspondence between Ramanujan-type formulas for $1/\pi$, and formulas for Dirichlet L -values. If we have an identity of the form

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(s)_n (\frac{1}{2})_n (1-s)_n}{n!^3} (a + bn) z^n,$$

where $(s)_n = \Gamma(s+n)/\Gamma(s)$, then under certain conditions we prove that

$$\sum_{n=1}^{\infty} \frac{n!^3}{(s)_n (\frac{1}{2})_n (1-s)_n} \frac{(a - bn)}{n^3} z^{-n}$$

reduces to Dirichlet L -values evaluated at 2. The two sums rarely converge at the same time, however divergent formulas make sense when they are interpreted as values of analytically continued hypergeometric functions. The same method also allows us to resolve certain values of the Epstein zeta function in terms of rapidly converging hypergeometric functions. The Epstein zeta functions were previously studied by Glasser and Zucker in [7].

1. INTRODUCTION

Quantities such as π^2 and the Dirichlet L -values are fundamental constants which appear in many areas of mathematics and physics. It is interesting to relate them to hypergeometric functions, which are important because of their applications in number theory. For instance, Apéry proved the irrationality of $\zeta(3)$ using a ${}_4F_3$ identity [6]. Ramanujan discovered many famous hypergeometric formulas for $1/\pi$. The following example [13]:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n}} \binom{2n}{n}^3 \left(\frac{1}{2} + 2n \right), \quad (1)$$

is connected to class number problems, and to the theory of complex multiplication [5], [6]. In this paper we describe identities which are closely related to Ramanujan's formulas. Our first example can be constructed by manipulating (1). Let $(1/2 + 2n) \mapsto (1/2 - 2n)$, flip the rest of the summand “upside-down”, insert a factor of $1/n^3$, and perform the summation for $n \geq 1$. Then we obtain a *companion*

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series identity:

$$8L_{-4}(2) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{6n}}{n^3 \binom{2n}{n}^3} \left(\frac{1}{2} - 2n \right). \quad (2)$$

As usual $L_{-4}(2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} \dots$ is Catalan's constant, $L_k(s) := \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s}$ denotes the general Dirichlet L -series, and $\chi_k(n) = \left(\frac{k}{n}\right)$ is the Jacobi symbol. Based on this example, we might expect that the same procedure should transform each of Ramanujan's formulas into identities involving Dirichlet L -values. We prove that this guess is correct when certain technical conditions are added. It is important to note that at least nine similar formulas already exist in the literature. The individual formulas were discovered piecemeal with computational techniques, and proved by diverse methods. We mention proofs due to Zeilberger [17], Guillera [8] [9] [11], and the Hessami-Pilehroods [12]. Sun also observed several identities from numerical experiments [14]. We give unified proofs of all of these results and conjectures in Theorem 3. We also show how to construct vast numbers of irrational formulas (such as (62) and the examples in Table 5), which were previously unknown. We describe our results in greater detail below.

Ramanujan identified seventeen formulas for $1/\pi$ [13]. His identities all have the following form:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n}{(1)_n^3} (a + bn) z^n, \quad (3)$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$. Each example has $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$, with (a, b, z) being parameterized by modular functions [5], [6]. When $s = \frac{1}{6}$, $z = \frac{1}{j(\tau)}$, where $j(\tau)$ is the j -invariant, and the expressions for a and b involve Eisenstein series. If we preserve the modular parameterizations for (a, b, z) , then the general *companion series* is given by

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n} \frac{(a - bn)}{n^3} z^{-n}. \quad (4)$$

When n is large, standard asymptotics show that $\frac{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n}{(1)_n^3} \approx \frac{\sin(\pi s)}{(\pi n)^{3/2}}$. It follows that (3) and (4) can only converge simultaneously if $|z| = 1$ (notice that (1) and (2) occur when $s = \frac{1}{2}$ and $(a, b, z) = (\frac{1}{2}, 2, -1)$). Divergent cases still make sense, provided that each divergent infinite series is replaced by an analytically-continued hypergeometric function. One of the main goals of this work, is to transform divergent formulas for $1/\pi$, into interesting convergent formulas for Dirichlet L -values.

Suppose that $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. Then Propositions 2 and 3 reduce many values of the companion series (4), to linear combinations of two Epstein zeta function and elementary constants. In general, once we fix the modular parameterizations for (a, b, z) in (4), then Propositions 2 and 3 harshly restrict the domain of the modular functions (see the constraints on equations (47) and (48)). This means there are fewer *potential* companion series evaluations, compared to the number of possible Ramanujan-type formulas coming from (3). Finally, if the linear combination of

Epstein zeta functions reduce to Dirichlet L -values, which is by no means automatic, then the companion series also reduces to Dirichlet L -values. Proofs are based upon a new idea called *completing the hypergeometric function*, which we outline in Section 3. The approach fails completely when $s = \frac{1}{6}$, and we describe the rationale for this failure at the end of Section 3. The Epstein zeta functions which appear have been studied by Glasser and Zucker [7]. Following their notation, define

$$S(A, B, C; t) := \sum_{(n,m) \neq (0,0)} \frac{1}{(An^2 + Bnm + Cm^2)^t}. \quad (5)$$

We demonstrate a calculation by proving (2). Set $q = -e^{-\pi\sqrt{2}}$ in (43). Then $(a, b, z) = (\frac{1}{2}, 2, -1)$. By equation (47), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{(1)_n^3}{(\frac{1}{2})_n^3} \left(\frac{1}{2} - 2n \right) = \frac{32\sqrt{2}}{\pi^2} (S(1, 0, 8; 2) - S(3, 4, 4; 2)).$$

Notice that $S(3, 4, 4; t)$ does not correspond to a reduced quadratic form ($C \geq A \geq |B|$), but it is possible to show that $S(3, 4, 4; t) = S(3, 2, 3; t)$. The key to completing the proof, is to reduce $S(A, B, C; t)$ to Dirichlet L -values. It is fortunate that this is a well-known problem. Let us briefly recall that quadratic forms with fixed discriminant $D = B^2 - 4AC$, are partitioned into equivalence classes under the action of $SL_2(\mathbb{Z})$. We say that quadratic forms of discriminant $D < 0$ have *one class per genus*, when disjoint classes of forms always represent disjoint sets of integers. Glasser and Zucker conjectured that $S(A, B, C; t)$ reduces to Dirichlet L -values, if and only if $An^2 + Bnm + Cm^2$ lives in a class of quadratic forms with one class per genus. Despite the fact that Zucker and Robertson discovered a few strange counterexamples to this conjecture [19], most evidence suggests that the original conjecture is “basically” correct. Every interesting companion series boils down to two values of $S(A, B, C; 2)$, and elementary constants. The proof of (2) follows from showing

$$\begin{aligned} S(1, 0, 8; 2) &= \frac{7\pi^2}{48} L_{-8}(2) + \frac{\pi^2}{8\sqrt{2}} L_{-4}(2), \\ S(3, 4, 4; 2) &= \frac{7\pi^2}{48} L_{-8}(2) - \frac{\pi^2}{8\sqrt{2}} L_{-4}(2). \end{aligned}$$

This type of reasoning explains all of the previously known companion series formulas, and all of the results in Theorems 3 and 4.

There are many instances where it is probably impossible to express $S(A, B, C; t)$ in terms of Dirichlet L -values. Then our method produces non-trivial hypergeometric formulas for $S(A, B, C; 2)$. For example, set $q = -e^{-\pi/3}$ in (43). After some work we obtain

$$\frac{48}{\pi^2} S(1, 0, 36; 2) = \frac{140}{27} L_{-4}(2) + \frac{13}{\sqrt{3}} L_{-3}(2) - \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{2})_n^3} \frac{(a - bn)}{n^3} z^{-n}, \quad (6)$$

where

$$\begin{aligned} z &= -8 \left(74977 + 40284r + 21644r^2 + 11629r^3 \right), \\ a &= \frac{1}{18} \left(1038 + 558r + 300r^2 + 161r^3 \right), \\ b &= \frac{1}{3} \left(387 + 208r + 112r^2 + 60r^3 \right), \end{aligned}$$

and $r = \sqrt[4]{12}$. Formula (6) converges very rapidly because $z \approx -2.4 \times 10^6$. The infinite series can either be expressed as a ${}_5F_4$ function, or as a linear combination of two ${}_4F_3$'s. In either case, this partially resolves a question of Zucker¹ and McPhedran [18], who asked whether or not $S(1, 0, 36; t)$ reduces to known quantities. See Section 5 for the proof of (6), and for additional examples.

2. REVIEW OF RAMANUJAN'S FORMULAS

We begin with a brief, but in-depth review of Ramanujan's formulas. Suppose that (3) holds for certain values of (a, b, z) and s . Let $y_0(z)$ denote the following ${}_3F_2$ function:

$$y_0(z) = {}_3F_2 \left(\begin{matrix} s, \frac{1}{2}, 1-s \\ 1, 1 \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n}{(1)_n^3} z^n. \quad (7)$$

We parameterize (a, b, z) in terms of q . Suppose that q and z are related by the differential equation:

$$\frac{dq}{dz} = \frac{q}{y_0(z)z\sqrt{1-z}}. \quad (8)$$

It is possible to express z in terms of q by integrating and then inverting (8). The inverse expressions are related to theta functions when $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ (we use (61) when $s = \frac{1}{2}$). The formulas for a and b are given by:

$$a = \frac{1}{\pi y_0(z)} \left(1 + \frac{\ln |q|}{y_0(z)} q \frac{dy_0(z)}{dq} \right), \quad b = -\frac{\ln |q|}{\pi} \sqrt{1-z}. \quad (9)$$

The parameterizations can be verified by substituting them into (3). It is a deep fact that (a, b, z) are *algebraic*, whenever $q = e^{2\pi i(x_1 + i\sqrt{|x_2|})}$ with $(x_1, x_2) \in \mathbb{Q}^2$, and $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$. The algebraic numbers are usually complicated, however rational evaluations occur in some instances.

Proposition 1. *Assume that (a, b, z) and q are related by (8) and (9). Suppose that $f(z)$ is a differentiable function, and let*

$$\phi_f(q) = \frac{f(z)}{y_0(z)}.$$

Then

$$af(z) + bz \frac{df(z)}{dz} = \frac{1}{\pi} \left(\phi_f(q) - \ln |q| q \frac{d\phi_f(q)}{dq} \right). \quad (10)$$

¹Zucker's dream is to resolve $S(1, 0, 36; t)$ in terms of Dirichlet L -values with complex characters.

Proof. From the right-hand side we have

$$\begin{aligned}
\frac{1}{\pi} \left(\phi_f(q) - \ln |q| q \frac{d\phi_f(q)}{dq} \right) &= \frac{1}{\pi} \left(\frac{f(z)}{y_0(z)} - \ln |q| q \frac{d}{dq} \frac{f(z)}{y_0(z)} \right) \\
&= \frac{1}{\pi} \frac{f(z)}{y_0(z)} - \ln |q| \frac{q}{\pi y_0^2(z)} \left(y_0(z) \frac{df(z)}{dz} - f(z) \frac{dy_0(z)}{dz} \right) \\
&= \frac{1}{\pi} \left(\frac{1}{y_0(z)} + \frac{\ln |q|}{y_0^2(z)} q \frac{dy_0(z)}{dz} \right) f(z) \\
&\quad - \left(\frac{\ln |q|}{\pi y_0(z)} \frac{q}{z} \frac{dz}{dq} \right) z \frac{df(z)}{dz} \\
&= af(z) + bz \frac{df(z)}{dz}.
\end{aligned}$$

The final step follows from (9). \square

Proposition 1 allows us to insert a factor of $(a + bn)$ into a power series. For example, if $f(z) = y_0(z)$, then $\phi_f(q) = 1$. We have

$$1 = \frac{1}{y_0(z)} \sum_{n=0}^{\infty} \frac{(s)_n (\frac{1}{2})_n (1-s)_n}{(1)_n^3} z^n.$$

By Proposition 1 this becomes

$$\frac{1}{\pi} \left(1 - \ln |q| q \frac{d}{dq} \right) \cdot 1 = \left(a + bz \frac{d}{dz} \right) \cdot \sum_{n=0}^{\infty} \frac{(s)_n (\frac{1}{2})_n (1-s)_n}{(1)_n^3} z^n,$$

hence

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(s)_n (\frac{1}{2})_n (1-s)_n}{(1)_n^3} (a + bn) z^n.$$

More difficult cases require us to expand $f(z)/y_0(z)$ in a q -series, before applying Proposition 1.

3. COMPLETING THE HYPERGEOMETRIC FUNCTION

In this section we introduce the idea of *completing a hypergeometric function*. Hypergeometric functions are typically defined by an infinite series, and analytically continued to a slit plane via integral formulas. To complete a hypergeometric function, let $n \mapsto n+x$ in the series definition, and extend the sum over $n \in \mathbb{Z}$. Consider $y_0(z)$, defined in (7), as an example. The completed version of $y_0(z)$ is a formal sum

$$\sum_{n \in \mathbb{Z}} \frac{(s)_{n+x} (\frac{1}{2})_{n+x} (1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x}, \quad (11)$$

which involves powers of z and z^{-1} . To avoid divergence issues, consider the positive ($n \geq 0$) and negative ($n < 0$) halves of the sum as hypergeometric functions. This

transforms (11) into a well-defined function:

$$Y_x(z) := z^x \frac{\left(\frac{1}{2}\right)_x (1-s)_x (s)_x}{(1)_x^3} {}_4F_3 \left(\begin{matrix} 1, \frac{1}{2} + x, 1 - s + x, s + x \\ 1 + x, 1 + x, 1 + x \end{matrix} \middle| z \right) \\ - \frac{2x^3 z^{x-1} \left(-\frac{1}{2}\right)_x (s-1)_x (-s)_x}{s(1-s) (1)_x^3} {}_4F_3 \left(\begin{matrix} 1, 1 - x, 1 - x, 1 - x \\ \frac{3}{2} - x, 2 - s - x, 1 + s - x \end{matrix} \middle| \frac{1}{z} \right) \quad (12)$$

which is certainly analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ (the ${}_4F_3$ functions and z^x have branch cuts on the real axis). From (11) it is obvious that $Y_x(z)$ is periodic in x :

$$Y_x(z) = Y_{x+1}(z).$$

This property extends to (12), because ${}_4F_3$ functions obey recurrences in their parameters, regardless of z . Below we prove that $Y_x(z)$ equals a trigonometric polynomial in x . This is the key result which enables us to sum up the companion series in Theorem 1.

Lemma 1. *Suppose that $s \in (0, 1)$ and $z \notin \{0, 1\}$. There exist functions $u := u(z)$ and $v := v(z)$ which are independent of x , such that*

$$Y_x(z) = y_0(z) \frac{e^{i\pi x} \sin^2 \pi s}{\cos \pi x (\cos^2 \pi x - \cos^2 \pi s)} (-u + (u + 1) \cos 2\pi x - iv \sin 2\pi x). \quad (13)$$

Proof. Consider the Picard-Fuchs operator which annihilates $y_0(z)$. Let

$$P := \left(z \frac{d}{dz} \right)^3 - z \left(z \frac{d}{dz} + \frac{1}{2} \right) \left(z \frac{d}{dz} + s \right) \left(z \frac{d}{dz} + 1 - s \right). \quad (14)$$

If convergence issues are ignored, then it is easy to show that P also annihilates (11). This allows us to extrapolate

$$PY_x(z) = 0. \quad (15)$$

It is possible to prove (15) using standard rules for differentiating hypergeometric functions, but we leave this as an exercise. Since P annihilates $Y_x(z)$, the function has the form:

$$Y_x(z) = m_0(x)y^{(0)}(z) + m_1(x)y^{(1)}(z) + m_2(x)y^{(2)}(z), \quad (16)$$

where each $y^{(i)}$ is a linearly independent solution of $Py = 0$. The linear independence property implies that $m_i(x) = m_i(x + 1)$ for all i (if the m_i 's are not periodic, then $Y_x(z) - Y_{x+1}(z) = 0$ leads to a linear dependence between $y^{(i)}$'s). We derive formulas for $m_i(x)$ below.

Suppose that $s \in (0, 1)$, and that z is not a singular point of $Y_x(z)$ (we exclude $z = 0$ and $z = 1$). Since $Y_x(z) = Y_{x+1}(z)$, we assume without loss of generality that $\operatorname{Re}(x) \in [0, 1)$. We claim that $Y_x(z)$ is meromorphic in x , with simple poles at $x \in \{s, \frac{1}{2}, 1 - s\}$. To prove this, first recall that ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z)$, is meromorphic with respect to each b_i , provided z is not a singular point [1]. Poles occur if $b_i \in \{0, -1, -2, \dots\}$. Since $(\operatorname{Re}(x), s) \in [0, 1) \times (0, 1)$, it is easy to check

that $\{1+x, \frac{3}{2}-x, 2-s-x, 1+s-x\} \cap \{0, -1, -2, \dots\} = \emptyset$, thus the ${}_4F_3$ functions in (12) do not contribute poles. Next observe

$$\frac{\left(-\frac{1}{2}\right)_x (s-1)_x (-s)_x}{(1)_x^3} = \frac{\Gamma(-\frac{1}{2}+x)\Gamma(s-1+x)\Gamma(-s+x)}{\Gamma(\frac{1}{2})\Gamma(-s)\Gamma(s-1)\Gamma^3(1+x)},$$

$$\frac{\left(\frac{1}{2}\right)_x (1-s)_x (s)_x}{(1)_x^3} = \frac{\Gamma(\frac{1}{2}+x)\Gamma(1-s+x)\Gamma(s+x)}{\Gamma(\frac{1}{2})\Gamma(1-s)\Gamma(s)\Gamma^3(1+x)}.$$

The first ratio of Pochhammer symbols contributes simple poles when $x \in \{s, \frac{1}{2}, 1-s\}$, and the second ratio of Pochhammer symbols is analytic for $(\operatorname{Re}(x), s) \in [0, 1) \times (0, 1)$. By the linear independence argument above, we conclude that $m_i(x)$ is at worst meromorphic with simple poles when $x \in \{s, \frac{1}{2}, 1-s\}$.

Now we show that $m_i(x) = O(|\operatorname{Im}(x)|^{-3/2})$ when $|\operatorname{Im}(x)|$ is sufficiently large. Let $z \in [\epsilon, 1-\epsilon]$, for some small $\epsilon > 0$. Note that $|z^x| = |z|^{\operatorname{Re}(x)} < 1$. Formula (12) becomes

$$|Y_x(z)| < \left| \frac{\left(\frac{1}{2}\right)_x (1-s)_x (s)_x}{(1)_x^3} {}_4F_3 \left(\begin{matrix} 1, \frac{1}{2}+x, 1-s+x, s+x \\ 1+x, 1+x, 1+x \end{matrix} \middle| z \right) - \frac{2x^3 z^{-1}}{s(1-s)} \frac{\left(-\frac{1}{2}\right)_x (s-1)_x (-s)_x}{(1)_x^3} {}_4F_3 \left(\begin{matrix} 1, 1-x, 1-x, 1-x \\ \frac{3}{2}-x, 2-s-x, 1+s-x \end{matrix} \middle| \frac{1}{z} \right) \right|.$$

The right-hand side of the inequality vanishes when $|\operatorname{Im}(x)| \mapsto \infty$. To see this, use the estimates

$${}_4F_3 \left(\begin{matrix} 1, \frac{1}{2}+x, 1-s+x, s+x \\ 1+x, 1+x, 1+x \end{matrix} \middle| z \right) \approx {}_1F_0 \left(\begin{matrix} 1 \\ \end{matrix} \middle| z \right) = \frac{1}{1-z}$$

$${}_4F_3 \left(\begin{matrix} 1, 1-x, 1-x, 1-x \\ \frac{3}{2}-x, 2-s-x, 1+s-x \end{matrix} \middle| \frac{1}{z} \right) \approx {}_1F_0 \left(\begin{matrix} 1 \\ \end{matrix} \middle| \frac{1}{z} \right) = \frac{z}{z-1},$$

$$\frac{(1-s)_x \left(\frac{1}{2}\right)_x (s)_x}{(1)_x^3} \approx \frac{\sin \pi s}{(\pi i \operatorname{Im}(x))^{3/2}},$$

$$\frac{2x^3}{s(1-s)} \frac{\left(-\frac{1}{2}\right)_x (s-1)_x (-s)_x}{(1)_x^3} \approx -\frac{\sin \pi s}{(\pi i \operatorname{Im}(x))^{3/2}},$$

which are valid when $|\operatorname{Im}(x)|$ is large. Thus if $|\operatorname{Im}(x)|$ is sufficiently large (which rules out the possibility of x lying in a neighborhood of the points $\{s, \frac{1}{2}, 1-s\}$), then $Y_x(z) = O(|\operatorname{Im}(x)|^{-3/2})$. The estimate holds uniformly for $z \in [\epsilon, 1-\epsilon]$, so a linear independence argument suffices to show that $m_i(x) = O(|\operatorname{Im}(x)|^{-3/2})$ for each i .

We have proved that $m_i(x)$ is periodic and meromorphic, with (possible) simple poles at $x \in \{s, \frac{1}{2}, 1-s\}$. If $|\operatorname{Im}(x)|$ is sufficiently large, then $m_i(x) = O(|\operatorname{Im}(x)|^{-3/2})$. We conclude that

$$e^{-i\pi x} \cos \pi x (\cos^2 \pi x - \cos^2 \pi s) m_i(x)$$

is analytic for $\operatorname{Re}(x) \in [0, 1)$. This new function has period 1, so it is also analytic on \mathbb{C} . The function is majorized by $O(|\operatorname{Im}(x)|^{-3/2} e^{4\pi|\operatorname{Im}(x)|})$ for $|\operatorname{Im}(x)|$ sufficiently

large. Therefore the function has a Fourier series which terminates:

$$e^{-i\pi x} \cos \pi x (\cos^2 \pi x - \cos^2 \pi s) m_i(x) = a_i^{(0)} + a_i^{(1)} \cos(2\pi x) + a_i^{(2)} \sin(2\pi x).$$

After collecting constants in (16), and noting that $Y_0(z) = y_0(z)$, we conclude that $Y_x(z)$ has the form given in (13). \square

Now let $y_x(z)$ denote the positive half ($n \geq 0$) of the completed hypergeometric function:

$$\begin{aligned} y_x(z) &:= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{n+x} (s)_{n+x} (1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x} \\ &= z^x \frac{(\frac{1}{2})_x (1-s)_x (s)_x}{(1)_x^3} {}_4F_3 \left(\begin{matrix} 1, \frac{1}{2} + x, 1-s+x, s+x \\ 1+x, 1+x, 1+x \end{matrix} \middle| z \right). \end{aligned} \quad (17)$$

The first author calls this an *extended hypergeometric series* [9]. Since $y_x(z)$ is analytic in a neighborhood of $x = 0$, we have a Maclaurin series of the form

$$\frac{y_x(z)}{y_0(z)} = 1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4), \quad (18)$$

where z and q are related by (8). Since $y_x(z)/y_0(z)$ is non-holomorphic in z , we expect each $\phi_i(q)$ to be non-holomorphic in q .

Theorem 1. *Assume that $s \in (0, 1)$, $z \notin \{0, 1\}$, and let $\phi_i(q)$ be as in (18). Then*

$$\begin{aligned} \frac{1}{\pi y_0(z)} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n (\frac{1}{2})_n (1-s)_n} \frac{z^{-n}}{n^3} \\ = \pi^2 i \csc^2(\pi s) - \frac{\pi}{3} (1 + 3 \csc^2(\pi s)) \phi_1(q) - i \phi_2(q) + \frac{1}{\pi} \phi_3(q). \end{aligned} \quad (19)$$

By Proposition 1, we also have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n (\frac{1}{2})_n (1-s)_n} \frac{(a-bn)}{n^3} z^{-n} \\ = \pi^2 i \csc^2(\pi s) - \frac{\pi}{3} (1 + 3 \csc^2(\pi s)) \left(\phi_1(q) - q \log |q| \frac{d\phi_1(q)}{dq} \right) \\ - i \left(\phi_2(q) - q \log |q| \frac{d\phi_2(q)}{dq} \right) + \frac{1}{\pi} \left(\phi_3(q) - q \log |q| \frac{d\phi_3(q)}{dq} \right). \end{aligned} \quad (20)$$

The sums in (19) and (20) diverge if $|z| < 1$, however the identities remain valid when ${}_4F_3$ and ${}_5F_4$ functions are substituted.

Proof. From (12) and (17) we see that

$$Y_x(z) = y_x(z) + O(x^3).$$

This is sufficient to determine u and v in (13). From (17) we find

$$\frac{y_x(z)}{y_0(z)} = 1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4).$$

By (13) we also have

$$\begin{aligned} \frac{Y_x(z)}{y_0(z)} &= 1 + i\pi(1 - 2v)x + \pi^2(-2 - 2u + 2v + \csc^2(\pi s))x^2 \\ &\quad - \frac{i\pi^3}{3}(5 + 6u - 4v + (-3 + 6v)\csc^2(\pi s))x^3 + O(x^4), \end{aligned} \quad (21)$$

where s and z satisfy the appropriate restrictions. The Taylor coefficients of $Y_x(z)$ and $y_x(z)$ agree up to order x^2 . This leads to a pair of equations

$$\begin{aligned} \phi_1(q) &= i\pi(1 - 2v) \\ \phi_2(q) &= \pi^2(-2 - 2u + 2v + \csc^2(\pi s)), \end{aligned}$$

from which it is easy to solve for u and v .

The companion series arises from the x^3 coefficient of $Y_x(z)$. By (12) and (17) we have

$$\begin{aligned} \frac{1}{y_0(z)} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n} \frac{z^{-n}}{n^3} &= \frac{1}{y_0(z)} \frac{2z^{-1}}{s(1-s)} {}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ \frac{3}{2}, 2-s, 1+s \end{matrix} \middle| \frac{1}{z}\right) \\ &= \lim_{x \rightarrow 0} \left(\frac{y_x(z) - Y_x(z)}{y_0(z) x^3} \right) \\ &= \phi_3(q) + \frac{i\pi^3}{3}(5 + 6u - 4v + (-3 + 6v)\csc^2(\pi s)). \end{aligned}$$

We recover (19) by eliminating u and v . □

Despite the fact that (19) and (20) hold for many values of s , we have only been able to evaluate $\phi_i(q)$ if $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. We prove formulas for $\phi_i(q)$ below.

Theorem 2. *Suppose that q lies in a neighborhood of zero. When $s = \frac{1}{2}$:*

$$\phi_1(q) = \ln q, \quad (22)$$

$$\phi_2(q) = \frac{1}{2} \ln^2 q + \frac{\pi^2}{2}, \quad (23)$$

$$\phi_3(q) = \frac{1}{6} \ln^3 q + \frac{\pi^2}{2} \ln q - 6\zeta(3) - 16 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^n + 4 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^{4n}. \quad (24)$$

When $s = \frac{1}{3}$:

$$\phi_1(q) = \ln q, \quad (25)$$

$$\phi_2(q) = \frac{1}{2} \ln^2 q + \frac{2\pi^2}{3}, \quad (26)$$

$$\phi_3(q) = \frac{1}{6} \ln^3 q + \frac{2\pi^2}{3} \ln q - 10\zeta(3) - 30 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^n + 10 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^{3n}. \quad (27)$$

When $s = \frac{1}{4}$:

$$\phi_1(q) = \ln q, \quad (28)$$

$$\phi_2(q) = \frac{1}{2} \ln^2 q + \pi^2, \quad (29)$$

$$\phi_3(q) = \frac{1}{6} \ln^3 q + \pi^2 \ln q - 20\zeta(3) - 80 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^n + 40 \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3} q^{2n}. \quad (30)$$

Proof. The essential idea is to apply the Picard-Fuchs operator which annihilates $y_0(z)$. Recall that P is defined in (14). It was proved in [10, Prop. 2.2], that

$$Py_x(z) = \frac{(1-s)_x \left(\frac{1}{2}\right)_x (s)_x}{(1)_x^3} z^x x^3 = x^3 + O(x^4). \quad (31)$$

When $x = 0$, we immediately obtain the homogeneous differential equation $Py_0(z) = 0$. If $y_x(z)$ is expanded in a Maclaurin series with respect to x , then by (18) we have $P(y_0(z)\phi_1(q)) = 0$ and $P(y_0(z)\phi_2(q)) = 0$. Appealing to [15, Lemma 1], we see that

$$\left(q \frac{d}{dq}\right)^3 \phi_1(q) = 0, \quad \left(q \frac{d}{dq}\right)^3 \phi_2(q) = 0, \quad (32)$$

and integrating gives

$$\phi_1(q) = \alpha_0 + \alpha_1 \ln q + \alpha_2 \ln^2 q, \quad (33)$$

$$\phi_2(q) = \beta_0 + \beta_1 \ln q + \beta_2 \ln^2 q, \quad (34)$$

where the α_i 's and β_i 's are undetermined constants. Examining the x^3 coefficient of $y_x(z)$, leads to the inhomogeneous differential equation $P[y_0(z)\phi_3(q)] = 1$. By [15, Lemma 1] and [10, Iden. 2.33], we find that

$$\left(q \frac{d}{dq}\right)^3 \phi_3(q) = \sqrt{1-z} y_0^2(z). \quad (35)$$

In order to solve (35), and to determine the constants in (33) and (34), it is necessary to specify the value of s .

Suppose that q lies in a neighborhood of zero. When $s = \frac{1}{2}$ we have $\sqrt{1-z} = 1 - 2\lambda(q)$, where $\lambda(q) = \theta_2^4(q)/\theta_3^4(q)$ is the elliptic lambda function [10, Sect. 2.5]. By standard theta function inversion formulas, we also have

$$y_0(z) = \theta_3^4(q). \quad (36)$$

Identity (36) does not hold for $|q| < 1$. For instance, if q is close to 1 we have to replace (36) with $y_0(z) = \frac{\log^2(q)}{\pi^2} \theta_3^4(q)$. For $|q|$ sufficiently small

$$\begin{aligned} y_0^2(z) \sqrt{1-z} &= \theta_3^8(q) - 2\theta_3^4(q)\theta_2^4(q) \\ &= 1 - 16 \sum_{n=1}^{\infty} \sigma_3(n) q^n + 16^2 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n}, \end{aligned}$$

where the second equality follows from [3, pg. 126, Entry 13]. Integrating (35) gives

$$\begin{aligned} \phi_3(q) = & \gamma_0 + \gamma_1 \ln q + \gamma_2 \ln^2 q + \frac{1}{6} \ln^3 q \\ & - 16 \sum_{n=1}^{\infty} \sigma_3(n) \frac{q^n}{n^3} + 4 \sum_{n=1}^{\infty} \sigma_3(n) \frac{q^{4n}}{n^3}, \end{aligned} \quad (37)$$

where the γ_i 's are constants.

There are nine constants left to calculate. Let q tend to zero in (18). Since z has a q -series of the form $z = 64q + O(q^2)$, it follows that $z \approx 64q$ when q approaches zero. In a similar manner we find that $y_0(z) \approx 1$. By (18) we have

$$\begin{aligned} q^{-x} y_x(z) &= q^{-x} y_0(z) (1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4)) \\ &\approx q^{-x} (1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4)). \end{aligned} \quad (38)$$

From the definition of $q^{-x} y_x(z)$, we calculate

$$\begin{aligned} q^{-x} y_x(z) &= q^{-x} z^x \frac{\left(\frac{1}{2}\right)_x^3}{(1)_x^3} \left(1 + \sum_{n=1}^{\infty} z^n \frac{\left(\frac{1}{2} + x\right)_n^3}{(1+x)_n^3}\right) \\ &\approx 64^x \frac{\left(\frac{1}{2}\right)_x^3}{(1)_x^3} (1 + 0) \end{aligned} \quad (39)$$

Compare the Maclaurin series coefficients of (38) and (39) in x , x^2 , and x^3 . Since (39) is holomorphic at $x = 0$, it follows that (38) is holomorphic at $x = 0$ as well. Since q tends to zero, this implies that the powers of $\log(q)$ must drop out of the series obtained from (38). Comparing coefficients then provides sufficiently many relations to determine the values of α_i , β_i , and γ_i explicitly. The cases when $s = \frac{1}{3}$ and $s = \frac{1}{4}$ require analogous arguments, using appropriate theta functions from [4]. \square

The method fails when $s = \frac{1}{6}$, because of our inability to solve (35). The calculation is difficult because Ramanujan's theory of signature-6 modular equations is incomplete, and as a result it seems to be impossible to find a nice q -series expansion for $\sqrt{1-z} y_0^2(z)$. Notice that (35) is equivalent to

$$\left(q \frac{d}{dq}\right)^3 \phi_3(q) = \frac{1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}}{\sqrt{1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}}}. \quad (40)$$

If we could obtain a reasonable expression for $\phi_3(q)$, then it might be possible to evaluate a companion series with $s = \frac{1}{6}$. Experimental searches failed to turn up any interesting identities, so we suspect that the task is impossible.

4. EXPLICIT FORMULAS

Now we prove companion series evaluations. Proposition 2 reduces every companion series to elementary constants and values of the following special function:

$$F(q) := -\frac{\log^3 |q|}{3\pi} + \frac{120}{\pi} \zeta(3) + \frac{240}{\pi} \sum_{j=1}^{\infty} \text{Li}_3(q^j) - \log |q| \text{Li}_2(q^j). \quad (41)$$

Notice that $F(q)$ is closely related to the elliptic trilogarithm [16]. Set $q = e^{2\pi i \tau}$, with $\tau = x + iy$, and $y > 0$. In Proposition 3 we prove

$$\text{Re}(F(q)) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2). \quad (42)$$

It is easy to see that $F(q)$ is real-valued if $q \in (-1, 1)$, so (42) becomes a formula for $F(q)$ whenever $x \in \mathbb{Z}/2$. Glasser and Zucker proved that $S(A, B, C; t)$ reduces to Dirichlet L -values quite often. This leads to 65 evaluations of $F(q)$, when $x = 0$ and $y^2 \in \mathbb{N}$. For instance, when $(x, y) = (0, \sqrt{7})$, we have

$$F(e^{-2\pi\sqrt{7}}) = 175\sqrt{7}L_{-7}(2).$$

Various additional values of $F(q)$ are provided in Table 1. The formulas in Theorems 3 and 4 are proved by evaluating linear combinations of $F(q)$'s.

Proposition 2. *Suppose that q lies in a neighborhood of zero. When $s = \frac{1}{2}$:*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a - bn)}{n^3} z^{-n} &= -\frac{1}{15}F(q) + \frac{1}{60}F(q^4) \\ &+ \frac{\log(q)^3}{6\pi} - \frac{\log(q)^2 \log |q|}{2\pi} + \frac{\log |q|^3}{3\pi} \\ &- \frac{i}{2} \log(q)^2 + i \log(q) \log |q| \\ &- \frac{5}{6}\pi \log(q) + \frac{5}{6}\pi \log |q| + \frac{i\pi^2}{2}. \end{aligned} \quad (43)$$

When $s = \frac{1}{3}$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n} \frac{(a - bn)}{n^3} z^{-n} &= -\frac{1}{8}F(q) + \frac{1}{24}F(q^3) \\ &+ \frac{\log^3(q)}{6\pi} - \frac{\log^2(q) \log |q|}{2\pi} + \frac{\log^3 |q|}{3\pi} \\ &- \frac{i}{2} \log^2(q) + i \log(q) \log |q| \\ &- \pi \log(q) + \pi \log |q| + \frac{2i\pi^2}{3}. \end{aligned} \quad (44)$$

When $s = \frac{1}{4}$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n} \frac{(a - bn)}{n^3} z^{-n} &= -\frac{1}{3}F(q) + \frac{1}{6}F(q^2) \\ &+ \frac{\log^3(q)}{6\pi} - \frac{\log^2(q) \log |q|}{2\pi} + \frac{\log^3 |q|}{3\pi} \\ &- \frac{1}{2}i \log^2(q) + i \log(q) \log |q| \\ &- \frac{4}{3}\pi \log(q) + \frac{4}{3}\pi \log |q| + i\pi^2. \end{aligned} \quad (45)$$

Proof. Proofs follow from combining Theorems 1 and 2. In particular, we obtain formulas (43) through (45), by substituting the results of Theorem 2 into (20). \square

Proposition 3. Let $q = e^{2\pi i\tau}$, with $\tau = x + iy$, and $y > 0$. Then

$$F(q) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2) + \frac{60i}{\pi^2} \sum_{\substack{n,k \\ n \neq 0}} \frac{(k + nx) ((k + nx)^2 + 3n^2y^2)}{n^3 ((k + nx)^2 + n^2y^2)^2}. \quad (46)$$

If $x \in \mathbb{Z}/2$ and $y > 0$, then

$$F(q) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2). \quad (47)$$

If $2x/(x^2 + y^2) \in \mathbb{Z}$ and $y > 0$, then

$$F(q) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2) + \frac{4i\pi^2}{3} x \left(\frac{x^2 + 3y^2}{(x^2 + y^2)^2} + x^2 + 3y^2 - 5 \right). \quad (48)$$

Proof. By (41) we obtain

$$\begin{aligned} F(q) &= \frac{8\pi^2}{3} (\text{Im } \tau)^3 + \frac{120}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{2}{n^3} \sum_{j=1}^{\infty} q^{jn} + \frac{4\pi \text{Im}(\tau)}{n^2} \sum_{j=1}^{\infty} j q^{jn} \right) \\ &= \frac{8\pi^2}{3} (\text{Im } \tau)^3 + \frac{120}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n^3} \frac{1 + q^n}{1 - q^n} + \frac{4\pi \text{Im}(\tau)}{n^2} \frac{q^n}{(1 - q^n)^2} \right) \\ &= \frac{8\pi^2}{3} (\text{Im } \tau)^3 + \frac{60}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{i \cot(\pi n\tau)}{n^3} - \frac{\pi \text{Im}(\tau) \csc^2(\pi n\tau)}{n^2} \right). \end{aligned}$$

Substitute the partial fractions decompositions:

$$\cot(\pi n\tau) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k + \tau n}, \quad \pi \csc^2(\pi n\tau) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(k + \tau n)^2},$$

to obtain

$$F(q) = \frac{8\pi^2}{3} (\text{Im } \tau)^3 + \frac{60}{\pi^2} \sum_{\substack{n,k=-\infty \\ n \neq 0}}^{\infty} \frac{i}{n^3(k + n\tau)} - \frac{\text{Im}(\tau)}{n^2(k + n\tau)^2}. \quad (49)$$

Formula (46) follows from setting $\tau = x + iy$, and then isolating the real and imaginary parts of the function. We complete the proof of (47) by noting that $F(q)$ is real valued whenever $x \in \mathbb{Z}/2$.

To complete the proof of (48) we need to evaluate the following sum:

$$T(x, y) := \sum_{\substack{n, k \\ n \neq 0}} \frac{(k + nx) ((k + nx)^2 + 3n^2 y^2)}{n^3 ((k + nx)^2 + n^2 y^2)^2}.$$

Extract the $k = 0$ term, to obtain

$$T(x, y) = \frac{\pi^4}{45} \frac{x(x^2 + 3y^2)}{(x^2 + y^2)^2} + \sum_{\substack{k \\ k \neq 0}} \sum_{\substack{n \\ n \neq 0}} \frac{(k + nx) ((k + nx)^2 + 3n^2 y^2)}{n^3 ((k + nx)^2 + n^2 y^2)^2}.$$

When $k \neq 0$ the inner sum can be evaluated by the residues method. **Mathematica** produces the following formula:

$$\begin{aligned} & \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(k + nx) ((k + nx)^2 + 3n^2 y^2)}{n^3 ((k + nx)^2 + n^2 y^2)^2} \\ &= - \frac{x(\pi^2 k^2 - 9y^2 - 3x^2)}{3k^4} \\ & \quad - \pi \sin \left(\frac{2\pi kx}{x^2 + y^2} \right) \frac{(x^2 + y^2) \left(\cosh^2 \frac{\pi ky}{x^2 + y^2} - \cos^2 \frac{\pi kx}{x^2 + y^2} \right) + k\pi y \sinh \frac{2k\pi y}{x^2 + y^2}}{2k^3 \left(\cosh^2 \frac{\pi ky}{x^2 + y^2} - \cos^2 \frac{\pi kx}{x^2 + y^2} \right)^2}. \end{aligned}$$

If $2x/(x^2 + y^2) \in \mathbb{Z}$, then the second term vanishes. Thus we are left with

$$\begin{aligned} T(x, y) &= \frac{\pi^4}{45} \frac{x(x^2 + 3y^2)}{(x^2 + y^2)^2} - \sum_{\substack{k \\ k \neq 0}} \frac{x(\pi^2 k^2 - 9y^2 - 3x^2)}{3k^4} \\ &= \frac{\pi^4}{45} x \left(\frac{x^2 + 3y^2}{(x^2 + y^2)^2} + x^2 + 3y^2 - 5 \right), \end{aligned}$$

and (48) follows. □

4.1. Convergent rational formulas. Now we prove rational, convergent, companion series formulas. Virtually all of these results have appeared in the literature before, although we believe this is the first unified treatment of all of the formulas. Equation (52) was proved by Zeilberger [17, Theorem 8]. Formulas (50), (51), (53) are due to Guillera [8], [9]. Equations (54) through (58) were conjectured by Sun using numerical experiments [14]. Formula (57) was subsequently proved by Guillera [11], and the Hessami-Pilehroods proved (58) [12]. Our strategy is to express each companion series in terms of $F(q)$'s, and then to evaluate $F(q)$ using properties of Epstein zeta functions. The hypergeometric-side of the formula also requires values of (a, b, z) . It is straight-forward, albeit tedious, to calculate those quantities. We summarize the values of (a, b, z) and the corresponding q 's in Table 2.

q	$F(q)$
$e^{-2\pi}$	$80L_{-4}(2)$
$e^{-2\pi\sqrt{2}}$	$80\sqrt{2}L_{-8}(2)$
$e^{-2\pi\sqrt{3}}$	$135\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{4}}$	$280L_{-4}(2)$
$e^{-2\pi\sqrt{5}}$	$100\sqrt{5}L_{-20}(2) + 96L_{-4}(2)$
$e^{-2\pi\sqrt{6}}$	$120\sqrt{6}L_{-24}(2) + 90\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{7}}$	$175\sqrt{7}L_{-7}(2)$
$e^{-2\pi\sqrt{8}}$	$280\sqrt{2}L_{-8}(2) + 240L_{-4}(2)$
$e^{-2\pi\sqrt{9}}$	$560L_{-4}(2) + 180\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{10}}$	$200\sqrt{10}L_{-40}(2) + 192\sqrt{2}L_{-8}(2)$
$e^{-2\pi\sqrt{12}}$	$480L_{-4}(2) + \frac{1035}{2}\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{13}}$	$260\sqrt{13}L_{-52}(2) + 480L_{-4}(2)$
$e^{-2\pi\sqrt{15}}$	$\frac{375}{2}\sqrt{15}L_{-15}(2) + 468\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{16}}$	$480\sqrt{2}L_{-8}(2) + 1100L_{-4}(2)$
$e^{-2\pi\sqrt{18}}$	$880\sqrt{2}L_{-8}(2) + 540\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{21}}$	$210\sqrt{21}L_{-84}(2) + 210\sqrt{7}L_{-7}(2) + 480L_{-4}(2) + 360\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{22}}$	$440\sqrt{22}L_{-88}(2) + 330\sqrt{11}L_{-11}(2)$
$e^{-2\pi\sqrt{24}}$	$420\sqrt{6}L_{-24}(2) + 480\sqrt{2}L_{-8}(2) + 720L_{-4}(2) + 495\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{25}}$	$480\sqrt{5}L_{-20}(2) + 2320L_{-4}(2)$
$e^{-2\pi\sqrt{28}}$	$\frac{1435}{2}\sqrt{7}L_{-7}(2) + 1920L_{-4}(2)$
$e^{-2\pi\sqrt{30}}$	$300\sqrt{30}L_{-120}(2) + 288\sqrt{6}L_{-24}(2) + 225\sqrt{15}L_{-15}(2) + 630\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{33}}$	$330\sqrt{33}L_{-132}(2) + 330\sqrt{11}L_{-11}(2) + 1440L_{-4}(2) + 630\sqrt{3}L_{-3}(2)$

TABLE 1. Select values of $F(q)$

Theorem 3. *The following formulas are true:*

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3 (4n-1)}{\left(\frac{1}{2}\right)_n^3 n^3} = 16L_{-4}(2), \quad (50)$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3 (3n-1)}{\left(\frac{1}{2}\right)_n^3 n^3} \frac{1}{2^{2n}} = \frac{\pi^2}{2}, \quad (51)$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3 (21n-8)}{\left(\frac{1}{2}\right)_n^3 n^3} \frac{1}{2^{6n}} = \frac{\pi^2}{6}, \quad (52)$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3 (3n-1)}{\left(\frac{1}{2}\right)_n^3} \frac{1}{n^3} \frac{1}{2^{3n}} = 2L_{-4}(2), \quad (53)$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^{2n} = \frac{\pi^2}{2}, \quad (54)$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^n = 8\pi^2, \quad (55)$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(15n-4)}{n^3} \frac{1}{4^n} = 27L_{-3}(2), \quad (56)$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n} \frac{(5n-1)}{n^3} \left(\frac{3}{4}\right)^{2n} = \frac{45}{2}L_{-3}(2), \quad (57)$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n} \frac{(35n-8)}{n^3} \left(\frac{3}{4}\right)^{4n} = 12\pi^2. \quad (58)$$

Proof. We begin by proving (50). Set $q = -e^{-\pi\sqrt{2}}$ in (43). We have $(a, b, z) = (\frac{1}{2}, 2, -1)$. The formula reduces to

$$\frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3 (4n-1)}{\left(\frac{1}{2}\right)_n^3} \frac{1}{n^3} = -\frac{1}{15}F\left(-e^{-\pi\sqrt{2}}\right) + \frac{1}{60}F\left(e^{-4\pi\sqrt{2}}\right).$$

Apply (47) to reduce the equation to

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)_n^3 (4n-1)}{\left(\frac{1}{2}\right)_n^3} \frac{1}{n^3} &= \frac{64\sqrt{2}}{\pi^2} S(1, 0, 8; 2) - \frac{4\sqrt{2}}{\pi^2} S\left(1, 1, \frac{3}{4}; 2\right) \\ &= \frac{64\sqrt{2}}{\pi^2} (S(1, 0, 8; 2) - S(3, 4, 4; 2)). \end{aligned}$$

Glasser and Zucker have evaluated $S(1, 0, 8; t)$ for all t [7]. Their method also applies to $S(3, 4, 4; t) = S(3, 2, 3; t)$. When $t = 2$, the formulas become

$$\begin{aligned} S(1, 0, 8; 2) &= \frac{7\pi^2}{48} L_{-8}(2) + \frac{\pi^2}{8\sqrt{2}} L_{-4}(2), \\ S(3, 4, 4; 2) &= \frac{7\pi^2}{48} L_{-8}(2) - \frac{\pi^2}{8\sqrt{2}} L_{-4}(2), \end{aligned}$$

and the result follows.

Next consider (51). Set $q = ie^{-\pi\sqrt{3}/2}$ in (43). We have $(a, b, z) = (-\frac{i}{2}, -\frac{3i}{2}, 4)$. The formula reduces to

$$\frac{i}{2} \sum_{n=1}^{\infty} \frac{(1)_n^3 (3n-1)}{\left(\frac{1}{2}\right)_n^3} \frac{1}{n^3} \frac{1}{2^{2n}} = \frac{3i\pi^2}{8} - \frac{1}{15}F\left(ie^{-\pi\sqrt{3}/2}\right) + \frac{1}{60}F\left(e^{-2\pi\sqrt{3}}\right).$$

Equate the imaginary parts, and apply (48). The equation reduces to

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(3n-1)}{n^3} \frac{1}{2^{2n}} = \frac{3\pi^2}{4} - \frac{2}{15} \operatorname{Im} F\left(ie^{-\pi\sqrt{3}/2}\right) \\ = \frac{\pi^2}{2}.$$

Next we prove (52). Set $q = e^{3\pi i/4}e^{-\pi\sqrt{7}/4}$ in (43). We have $(a, b, z) = (-2i, -\frac{21i}{4}, 64)$. The formula reduces to

$$\frac{i}{4} \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(21n-8)}{n^3} \frac{1}{2^{6n}} = \frac{9\pi^2 i}{64} - \frac{1}{15} F\left(e^{3\pi i/4}e^{-\pi\sqrt{7}/4}\right) + \frac{1}{60} F\left(-e^{-\pi\sqrt{7}}\right).$$

Equate the imaginary parts, then apply (48). We obtain

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(21n-8)}{n^3} \frac{1}{2^{6n}} = \frac{9\pi^2}{16} - \frac{4}{15} \operatorname{Im} F\left(e^{3\pi i/4}e^{-\pi\sqrt{7}/4}\right) \\ = \frac{\pi^2}{6}.$$

Next consider (53). Set $q = -e^{-\pi}$ in (43). We have $(a, b, z) = (1, 3, -8)$. The formula reduces to

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(3n-1)}{n^3} \frac{(-1)^{n+1}}{2^{3n}} = -\frac{1}{15} F(-e^{-\pi}) + \frac{1}{60} F(e^{-4\pi}).$$

Apply (47) to obtain

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(3n-1)}{n^3} \frac{(-1)^{n+1}}{2^{3n}} = -\frac{1}{\pi^2} S\left(1, 1, \frac{1}{2}; 2\right) + \frac{16}{\pi^2} S(1, 0, 4; 2) \\ = 2L_{-4}(2).$$

In the final step we used $S(1, 0, 4; 2) = \frac{7\pi^2}{24} L_{-4}(2)$, and $S\left(1, 1, \frac{1}{2}; 2\right) = 4S(2, 2, 1; 2) = 4S(1, 0, 1; 2) = \frac{8\pi^2}{3} L_{-4}(2)$. Both of these evaluations follow from the results of Glasser and Zucker [7].

Now consider (54). Set $q = e^{2\pi i/3}e^{-2\pi\sqrt{2}/3}$ in (44). We have $(a, b, z) = (-i, -\frac{10i}{3}, \frac{27}{2})$. The formula reduces to

$$\frac{i}{3} \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^n = \frac{26\pi^2 i}{81} - \frac{1}{8} F\left(e^{2\pi i/3}e^{-2\pi\sqrt{2}/3}\right) + \frac{1}{24} F\left(e^{-2\pi\sqrt{2}}\right).$$

Take imaginary parts, then apply (48). We obtain

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^n = \frac{26\pi^2}{27} - \frac{3}{8} \operatorname{Im} F\left(e^{2\pi i/3}e^{-2\pi\sqrt{2}/3}\right) \\ = \frac{\pi^2}{2}.$$

Next we prove (55). Set $q = e^{\pi i/3} e^{-\pi\sqrt{11}/3}$ in (44). We have $(a, b, z) = (-\frac{i}{4}, -\frac{11i}{12}, \frac{27}{16})$. The formula reduces to

$$\frac{i}{12} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^n = \frac{64\pi^2 i}{81} - \frac{1}{8} F\left(e^{\pi i/3} e^{-\pi\sqrt{11}/3}\right) + \frac{1}{24} F\left(-e^{-\pi\sqrt{11}}\right).$$

Take imaginary parts, then apply (48). We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^n &= \frac{256\pi^2}{27} - \frac{3}{2} \operatorname{Im} F\left(e^{\pi i/3} e^{-\pi\sqrt{11}/3}\right) \\ &= 8\pi^2. \end{aligned}$$

Now prove (56). Set $q = -e^{-\pi\sqrt{15}/3}$ in (44). We have $(a, b, z) = (\frac{4}{3\sqrt{3}}, \frac{5}{\sqrt{3}}, -4)$. The formula reduces to

$$\frac{1}{3\sqrt{3}} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(15n-4)}{n^3} \frac{(-1)^{n+1}}{4^n} = -\frac{1}{8} F\left(-e^{-\pi\sqrt{15}/3}\right) + \frac{1}{24} F\left(-e^{-\pi\sqrt{15}}\right).$$

Apply (47) to obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(15n-4)}{n^3} \frac{(-1)^{n+1}}{4^n} &= -\frac{75\sqrt{5}}{8\pi^2} S\left(1, 1, \frac{2}{3}; 2\right) + \frac{675\sqrt{5}}{8\pi^2} S(1, 1, 4; 2) \\ &= \frac{675\sqrt{5}}{8\pi^2} (S(1, 1, 4; 2) - S(2, 3, 3; 2)). \end{aligned}$$

Glasser and Zucker have calculated $S(1, 1, 4; t)$ for all t [7]. Their method also applies to $S(2, 3, 3; t) = S(2, 1, 2; t)$. When $t = 2$ the formulas reduce to

$$\begin{aligned} S(1, 1, 4; 2) &= \frac{\pi^2}{6} L_{-15}(2) + \frac{4\pi^2}{25\sqrt{5}} L_{-3}(2), \\ S(2, 3, 3; 2) &= \frac{\pi^2}{6} L_{-15}(2) - \frac{4\pi^2}{25\sqrt{5}} L_{-3}(2), \end{aligned}$$

and (56) follows.

Next we prove (57). Set $q = -e^{-\pi\sqrt{3}}$ in (45). We have $(a, b, z) = (\frac{1}{\sqrt{3}}, \frac{5}{\sqrt{3}}, -\frac{16}{9})$. The formula reduces to

$$\frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n} \frac{(5n-1)}{n^3} (-1)^{n+1} \left(\frac{3}{4}\right)^{2n} = -\frac{1}{3} F\left(-e^{-\pi\sqrt{3}}\right) + \frac{1}{6} F\left(e^{-2\pi\sqrt{3}}\right).$$

By (47), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n} \frac{(5n-1)}{n^3} (-1)^{n+1} \left(\frac{3}{4}\right)^{2n} &= -\frac{45}{\pi^2} S(1, 1, 1; 2) + \frac{180}{\pi^2} S(1, 0, 3; 2) \\ &= \frac{45}{2} L_{-3}(2). \end{aligned}$$

Glasser and Zucker proved that $S(1, 0, 3; 2) = \frac{3\pi^2}{8} L_{-3}(2)$, and $S(1, 1, 1; 2) = \pi^2 L_{-3}(2)$ [7].

s	q	a	b	z
$\frac{1}{2}$	$-e^{-\pi\sqrt{2}}$	$\frac{1}{2}$	2	-1
$\frac{1}{2}$	$ie^{-\pi\sqrt{3}/2}$	$-\frac{i}{2}$	$-\frac{3i}{2}$	4
$\frac{1}{2}$	$e^{3\pi i/4}e^{-\pi\sqrt{7}/4}$	$-2i$	$-\frac{21i}{4}$	64
$\frac{1}{2}$	$-e^{-\pi}$	1	3	-8
$\frac{1}{3}$	$e^{2\pi i/3}e^{-2\pi\sqrt{2}/3}$	$-i$	$-\frac{10i}{3}$	$\frac{27}{2}$
$\frac{1}{3}$	$e^{\pi i/3}e^{-\pi\sqrt{11}/3}$	$-\frac{i}{4}$	$-\frac{11i}{12}$	$\frac{27}{16}$
$\frac{1}{3}$	$-e^{-\pi\sqrt{15}/3}$	$\frac{4}{3\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	-4
$\frac{1}{4}$	$-e^{-\pi\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	$-\frac{16}{9}$
$\frac{1}{4}$	$ie^{-\pi\sqrt{7}/2}$	$-\frac{4i}{9}$	$-\frac{35i}{18}$	$\frac{256}{81}$

TABLE 2. Values of (a, b, z) in Theorem 3

Finally prove (58). Set $q = ie^{-\pi\sqrt{7}/2}$ in (45). We have $(a, b, z) = (-\frac{4i}{9}, -\frac{35i}{18}, \frac{256}{81})$. The formula reduces to

$$\frac{i}{18} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n} \frac{(35n-8)}{n^3} \left(\frac{3}{4}\right)^{4n} = \frac{7\pi^2 i}{8} - \frac{1}{3} F\left(ie^{-\pi\sqrt{7}/2}\right) + \frac{1}{6} F\left(-e^{-\pi\sqrt{7}}\right).$$

Take the imaginary part, then apply (48). We obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n} \frac{(35n-8)}{n^3} \left(\frac{3}{4}\right)^{4n} &= \frac{63\pi^2}{4} - 6 \operatorname{Im} F\left(ie^{-\pi\sqrt{7}/2}\right) \\ &= 12\pi^2. \end{aligned}$$

□

Table 2 summarizes the values of (a, b, z) and q in Theorem 3. These values also lead to divergent formulas for $1/\pi$. For instance, when $s = \frac{1}{3}$ and $(a, b, z) =$

$\left(\frac{4}{3\sqrt{3}}, \frac{5}{\sqrt{3}}, -4\right)$, we obtain (56), and

$$\frac{1}{\pi} = \frac{4}{3\sqrt{3}} {}_4F_3\left(\begin{matrix} \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{19}{15} \\ 1, 1, \frac{4}{15} \end{matrix} \middle| -4\right).$$

The right-hand side equals .3183098..., which agrees perfectly with the expected numerical value of $1/\pi$.

4.2. Divergent rational formulas. Next we examine divergent hypergeometric formulas for Dirichlet L -values. These are companions to the convergent formulas for $1/\pi$. Since the identities have $|z| < 1$, we have substituted a ${}_5F_4$ function for the divergent companion series:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n} \frac{(a-bn)}{n^3} z^{-n} \\ = \frac{2(a-b)}{s(1-s)z} {}_5F_4\left(\begin{matrix} 1, 1, 1, 1, 2 - \frac{a}{b} \\ \frac{3}{2}, 1+s, 2-s, 1 - \frac{a}{b} \end{matrix} \middle| z^{-1}\right). \end{aligned} \quad (59)$$

The ${}_5F_4$ function has a branch cut on the interval $[1, \infty)$ [1]. When z^{-1} lies on the branch cut, the function takes a complex value. The real part of the function is uniquely defined, but the sign of the imaginary part depends on how we approach the branch cut. We use the same computational method as `Mathematica` 8, namely when $z^{-1} \in [1, \infty)$, we define ${}_5F_4(\cdots | z^{-1}) = \lim_{\delta \rightarrow 0} {}_5F_4(\cdots | z^{-1} - i\delta)$.

Theorem 4. *The following identity holds:*

$$\frac{2(a-b)}{s(1-s)z} {}_5F_4\left(\begin{matrix} 1, 1, 1, 1, 2 - \frac{a}{b} \\ \frac{3}{2}, 1+s, 2-s, 1 - \frac{a}{b} \end{matrix} \middle| z^{-1}\right) = L(2), \quad (60)$$

for the values of s , (a, b, z) , and $L(2)$ in Tables 3 and 4.

Proof. Proofs are the same as in Theorem 3, so we only consider one example in detail. Set $q = e^{-\pi\sqrt{7}}$ in (43). By Table 4, we have $s = \frac{1}{2}$ and $(a, b, z) = \left(\frac{5}{16}, \frac{21}{8}, \frac{1}{64}\right)$. Applying (47) and then (59), reduces the formula reduces to

$$\begin{aligned} -1184 {}_5F_4\left(\begin{matrix} 1, 1, 1, 1, \frac{79}{42} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{37}{42} \end{matrix} \middle| 64\right) &= 4i\pi^2 - \frac{1}{15}F\left(e^{-\pi\sqrt{7}}\right) + \frac{1}{60}F\left(e^{-4\pi\sqrt{7}}\right) \\ &= 4i\pi^2 - \frac{112\sqrt{7}}{\pi^2}(S(4, 0, 7; 2) - S(1, 0, 28; 2)). \end{aligned}$$

By the results of Glasser and Zucker [7], we obtain

$$\begin{aligned} S(1, 0, 28; 2) &= \frac{41\pi^2}{384}L_{-7}(2) + \frac{2\pi^2}{7\sqrt{7}}L_{-4}(2), \\ S(4, 0, 7; 2) &= \frac{41\pi^2}{384}L_{-7}(2) - \frac{2\pi^2}{7\sqrt{7}}L_{-4}(2), \end{aligned}$$

and we recover the value of $L(2)$ in Table 4. After simplifying, we find that

$${}_5F_4\left(\begin{matrix} 1, 1, 1, 1, \frac{79}{42} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{37}{42} \end{matrix} \middle| 64\right) = -\frac{2}{37}L_{-4}(2) - \frac{1}{296}\pi^2 i.$$

All of the formulas in Tables 3 and 4 follow from analogous arguments. \square

4.3. Irrational formulas. We emphasize that the *vast majority* of companion series formulas involve irrational values of (a, b, z) . Consider the narrow class of formulas which arises from setting $q = e^{-2\pi\sqrt{v}}$ in (45). The companion series with $s = \frac{1}{4}$ reduces to a linear combination of $S(1, 0, v; 2)$, $S(1, 0, 4v; 2)$, and elementary constants. There are 24 values of $v \in \mathbb{N}$, for which both sums reduces to Dirichlet L -values [7]. The $v = 1$ case produces a rational, albeit divergent, companion series (Theorem 4 with $s = \frac{1}{4}$ and $(a, b, z) = (\frac{2}{9}, \frac{14}{9}, \frac{32}{81})$). The other 23 choices lead to formulas with complicated algebraic values of (a, b, z) . While it is possible to determine those numbers from modular equations, it is usually much easier to use a computer. Formulas (8) and (9) are rather unwieldy for computational purposes, so we found it convenient to use theta functions. Suppose that $s = \frac{1}{2}$, and that q lies in a neighborhood of zero. Then

$$\begin{aligned} z &= 4 \frac{\theta_3^4(-q)}{\theta_3^4(q)} \left(1 - \frac{\theta_3^4(-q)}{\theta_3^4(q)} \right), \\ a &= \frac{1}{\pi \theta_3^4(q)} \left(1 + \frac{8 \log |q|}{\theta_3(q)} \sum_{n=1}^{\infty} n^2 q^{n^2} \right), \\ b &= \frac{\log |q|}{\pi} \left(1 - 2 \frac{\theta_3^4(-q)}{\theta_3^4(q)} \right), \end{aligned} \tag{61}$$

where

$$\theta_3(q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}.$$

More complicated formulas are required if $s \in \{\frac{1}{3}, \frac{1}{4}\}$.

To give an example of an irrational formula, set $q = e^{9\pi i/8} e^{-\pi\sqrt{15}/8}$ in (43). We calculate $(a, b, z) \approx (11.09i, 26.54i, 3006.63)$. The PSLQ algorithm returns the following polynomials:

$$\begin{aligned} 0 &= 1 - 11ia + a^2, \\ 0 &= 495 - 1680ib + 64b^2, \\ 0 &= 4096 - 3008z + z^2. \end{aligned}$$

Therefore $(a, b, z) = \left(\frac{1}{2}i(11 + 5\sqrt{5}), \frac{3}{8}i(35 + 16\sqrt{5}), \frac{1}{4}(1 + \sqrt{5})^8 \right)$. After simplifying with (48), we arrive at the following identity:

$$\frac{\pi^2}{30} = \sum_{n=1}^{\infty} \frac{3(35 + 16\sqrt{5})n - 4(11 + 5\sqrt{5})}{n^3 \binom{2n}{n}^3} \left(\frac{\sqrt{5} - 1}{2} \right)^{8n}. \tag{62}$$

s	q	a	b	$z < 0$	$L(2)$
$\frac{1}{2}$	$-e^{-\pi\sqrt{2}}$	$\frac{1}{2}$	$\frac{4}{2}$	-1	$8L_{-4}(2)$
$\frac{1}{2}$	$-e^{-\pi\sqrt{4}}$	$\frac{1}{2\sqrt{2}}$	$\frac{6}{2\sqrt{2}}$	$-\frac{1}{8}$	$16\sqrt{2}L_{-8}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{9/3}}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$-\frac{9}{16}$	$10\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{17/3}}$	$\frac{7}{12\sqrt{3}}$	$\frac{51}{12\sqrt{3}}$	$-\frac{1}{16}$	$30\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{25/3}}$	$\frac{\sqrt{15}}{12}$	$\frac{9\sqrt{15}}{12}$	$-\frac{1}{80}$	$15\sqrt{15}L_{-15}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{41/3}}$	$\frac{106}{192\sqrt{3}}$	$\frac{1230}{192\sqrt{3}}$	$-\frac{1}{2^{10}}$	$120\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{49/3}}$	$\frac{26\sqrt{7}}{216}$	$\frac{330\sqrt{7}}{216}$	$-\frac{1}{3024}$	$70\sqrt{7}L_{-7}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{89/3}}$	$\frac{827}{1500\sqrt{3}}$	$\frac{14151}{1500\sqrt{3}}$	$-\frac{1}{500^2}$	$390\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{5}}$	$\frac{3}{8}$	$\frac{20}{8}$	$-\frac{1}{4}$	$32L_{-4}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{7}}$	$\frac{8}{9\sqrt{7}}$	$\frac{65}{9\sqrt{7}}$	$-\frac{16^2}{63^2}$	$\frac{35}{2}\sqrt{7}L_{-7}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{9}}$	$\frac{3\sqrt{3}}{16}$	$\frac{28\sqrt{3}}{16}$	$-\frac{1}{48}$	$60\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{13}}$	$\frac{23}{72}$	$\frac{260}{72}$	$-\frac{1}{18^2}$	$160L_{-4}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{25}}$	$\frac{41\sqrt{5}}{288}$	$\frac{644\sqrt{5}}{288}$	$-\frac{1}{5 \cdot 72^2}$	$160\sqrt{5}L_{-20}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{37}}$	$\frac{1123}{3528}$	$\frac{21460}{3528}$	$-\frac{1}{882^2}$	$800L_{-4}(2)$

TABLE 3. Values of (a, b, z) with $z < 0$ in Theorem 4

s	q	a	b	$z > 0$	$L(2)$
$\frac{1}{2}$	$e^{-\pi\sqrt{3}}$	$\frac{1}{4}$	$\frac{6}{4}$	$\frac{1}{4}$	$16L_{-4}(2) + 2\pi^2i$
$\frac{1}{2}$	$e^{-\pi\sqrt{7}}$	$\frac{5}{16}$	$\frac{42}{16}$	$\frac{1}{64}$	$64L_{-4}(2) + 4\pi^2i$
$\frac{1}{3}$	$e^{-\pi\sqrt{8/3}}$	$\frac{1}{3\sqrt{3}}$	$\frac{6}{3\sqrt{3}}$	$\frac{1}{2}$	$\frac{15}{2}\sqrt{3}L_{-3}(2) + 2\pi^2i$
$\frac{1}{3}$	$e^{-\pi\sqrt{16/3}}$	$\frac{8}{27}$	$\frac{60}{27}$	$\frac{2}{27}$	$40L_{-4}(2) + \frac{10}{3}\pi^2i$
$\frac{1}{3}$	$e^{-\pi\sqrt{20/3}}$	$\frac{8}{15\sqrt{3}}$	$\frac{66}{15\sqrt{3}}$	$\frac{4}{125}$	$39\sqrt{3}L_{-3}(2) + 4\pi^2i$
$\frac{1}{4}$	$e^{-2\pi}$	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{32}{81}$	$20L_{-4}(2) + 3\pi^2i$
$\frac{1}{4}$	$e^{-\pi\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$	$\frac{8}{2\sqrt{3}}$	$\frac{1}{9}$	$30\sqrt{3}L_{-3}(2) + 4\pi^2i$
$\frac{1}{4}$	$e^{-\pi\sqrt{10}}$	$\frac{4}{9\sqrt{2}}$	$\frac{40}{9\sqrt{2}}$	$\frac{1}{81}$	$64\sqrt{2}L_{-8}(2) + 6\pi^2i$
$\frac{1}{4}$	$e^{-\pi\sqrt{18}}$	$\frac{27}{49\sqrt{3}}$	$\frac{360}{49\sqrt{3}}$	$\frac{1}{74}$	$180\sqrt{3}L_{-3}(2) + 10\pi^2i$
$\frac{1}{4}$	$e^{-\pi\sqrt{22}}$	$\frac{19}{18\sqrt{11}}$	$\frac{280}{18\sqrt{11}}$	$\frac{1}{99^2}$	$110\sqrt{11}L_{-11}(2) + 12\pi^2i$
$\frac{1}{4}$	$e^{-\pi\sqrt{58}}$	$\frac{4412}{9801\sqrt{2}}$	$\frac{105560}{9801\sqrt{2}}$	$\frac{1}{99^4}$	$960\sqrt{2}L_{-8}(2) + 30\pi^2i$

TABLE 4. Values of (a, b, z) with $z > 0$ in Theorem 4

This should be compared to Ramanujan's irrational formula for $1/\pi$, since both formulas involve powers of the golden ratio [13]. Table 5 contains many additional irrational formulas.

5. IRREDUCIBLE VALUES OF $S(A, B, C; 2)$

Irreducible values of $S(A, B, C; 2)$ occur when the quadratic form $An^2 + Bnm + Cm^2$ fails the one class per genus test. Apart from a few oddball cases, it is probably impossible to reduce these sums to Dirichlet L -functions [19]. In this section, we prove that it is still possible to express some irreducible values of $S(A, B, C; 2)$ in

s	q	a	b	$ z > 1$	Value of equation (4)
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{2}}{2}}$	$\frac{3+2\sqrt{2}}{2}$	$\frac{8+5\sqrt{2}}{2}$	$\frac{-8}{(\sqrt{2}-1)^3}$	$2L_{-4}(2) - \sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\frac{\pi}{2}}$	$\frac{14+10\sqrt{2}}{2}$	$\frac{33+24\sqrt{2}}{2}$	$\frac{-16\sqrt{2}}{(\sqrt{2}-1)^6}$	$-\frac{13}{4}L_{-4}(2) + 2\sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{2}}{3}}$	$\frac{59+24\sqrt{6}}{6}$	$\frac{140+56\sqrt{6}}{6}$	$\frac{-1}{(5-2\sqrt{6})^4}$	$\frac{136}{9}L_{-4}(2) - \frac{16}{3}\sqrt{6}L_{-24}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{2\sqrt{3}}{3}}$	$\frac{3\sqrt{6}+7\sqrt{2}}{24}$	$\frac{6\sqrt{6}+30\sqrt{2}}{24}$	$\frac{-1}{2(\sqrt{3}-1)^6}$	$16\sqrt{2}L_{-8}(2) - 8\sqrt{6}L_{-24}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{6}}{3}}$	$\frac{5+4\sqrt{2}}{6}$	$\frac{12+12\sqrt{2}}{6}$	$\frac{-1}{(\sqrt{2}-1)^4}$	$-8L_{-4}(2) + \frac{16}{3}\sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{10}}{5}}$	$\frac{23+10\sqrt{5}}{10}$	$\frac{60+24\sqrt{5}}{10}$	$\frac{-1}{(\sqrt{5}-2)^4}$	$\frac{56}{5}L_{-4}(2) - 4\sqrt{5}L_{-20}(2)$
$\frac{1}{2}$	$e^{\frac{9\pi i}{8}}e^{-\pi\frac{\sqrt{15}}{8}}$	$\frac{4(11+5\sqrt{5})}{8}i$	$\frac{3(35+16\sqrt{5})}{8}i$	$\frac{2^{14}}{(\sqrt{5}-1)^8}$	$-\frac{1}{240}\pi^2i$
$\frac{1}{3}$	$-e^{-\pi\frac{\sqrt{21}}{3}}$	$\frac{10+7\sqrt{7}}{54}$	$\frac{21+39\sqrt{7}}{54}$	$\frac{-1}{26\sqrt{7}-68}$	$-20L_{-4}(2) + \frac{35}{4}\sqrt{7}L_{-7}(2)$
$\frac{1}{4}$	$-e^{-\pi\frac{\sqrt{21}}{3}}$	$\frac{27+20\sqrt{3}}{72}$	$\frac{84+112\sqrt{3}}{72}$	$\frac{-1}{(42-24\sqrt{3})^2}$	$-\frac{160}{3}L_{-4}(2) + 40\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\frac{3\pi\sqrt{5}}{5}}$	$\frac{3987+2124\sqrt{3}}{4840}$	$\frac{19380+7440\sqrt{3}}{4840}$	$\frac{-1}{(680\sqrt{3}-1178)^2}$	$\frac{544}{5}L_{-4}(2) - 72\sqrt{3}L_{-3}(2)$

TABLE 5. Select convergent irrational companion series evaluations.

terms of hypergeometric functions. Propositions 2 and 3 reduce every interesting companion series to two values of $S(A, B, C; 2)$. Sometimes it is possible to select q , so that one sum reduces to Dirichlet L -values, and one sum does not. Sometimes both values of $S(A, B, C; 2)$ are irreducible, but one of them can be eliminated by finding a multi-term linear dependence with Dirichlet L -functions.

To make a first attempt at finding a formula, set $q = e^{-3\pi}$ in (43). Then $s = \frac{1}{2}$ and $(a, b, z) = (\frac{1}{4}(18r - 5r^3), 12r - 3r^3, (7 + 4\sqrt{3})^{-2})$, where $r = \sqrt[4]{12}$. By (47), the companion series equals a linear combination of $S(1, 0, 36; 2)$, $S(4, 0, 9; 2)$ and elementary constants. We eliminate $S(4, 0, 9; 2)$ with a result from [18]:

$$\begin{aligned}
S(1, 0, 36; t) + S(4, 0, 9; t) &= (1 - 2^{-t} + 2^{1-2t}) (1 + 3^{1-2t}) L_1(t) L_{-4}(t) \\
&\quad + (1 + 2^{-t} + 2^{1-2t}) L_{-3}(t) L_{12}(t).
\end{aligned} \tag{63}$$

After noting that $L_1(2) = \frac{\pi^2}{6}$ and $L_{12}(2) = \frac{\pi^2}{6\sqrt{3}}$, we obtain a divergent formula:

$$\begin{aligned} \frac{2}{\pi^2} S(1, 0, 36; 2) &= \frac{49}{18^2} L_{-4}(2) + \frac{11}{48\sqrt{3}} L_{-3}(2) \\ &\quad - \left(\frac{161 + 93\sqrt{3}}{18^4 \sqrt{12}} \right) \operatorname{Re} \left[{}_5F_4 \left(\begin{matrix} 1, 1, 1, 1, \frac{21+\sqrt{3}}{12} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{9+\sqrt{3}}{12} \end{matrix} \middle| (7 + 4\sqrt{3})^2 \right) \right]. \end{aligned}$$

Many additional divergent formulas exist. We consider these formulas disappointing, because they appear to be quite useless. Rapidly converging formulas are more exciting, but trickier to produce.

Consider the restriction on q imposed in Proposition 2. To obtain an $s = \frac{1}{2}$ companion series from (43), we must select q to lie in a neighborhood of zero. Unwinding the proof of Theorem 2, shows that we can only select values of q for which

$$\theta_3^4(q) = {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4 \frac{\theta_3^4(-q)}{\theta_3^4(q)} \left(1 - \frac{\theta_3^4(-q)}{\theta_3^4(q)} \right) \right)$$

holds (similar restriction exist when $s = \frac{1}{3}$ and $s = \frac{1}{4}$). This constraint implies that the allowable values on the real axis are $q \in (-1, e^{-\pi})$. If $q \in (-e^{-\pi\sqrt{2}}, e^{-\pi})$ then $|z| < 1$, and the companion series diverges. On the other hand, if $q \in (-1, -e^{-\pi\sqrt{2}})$ then $|z| > 1$, and we obtain convergent formulas. Suppose that $q = e^{2\pi i(\frac{1}{2} + iy)}$, so that q lives on the negative real axis. Then by (47) we find

$$\begin{aligned} F(q) &= F(-e^{-2\pi y}) = \frac{120y^3}{\pi^2} S \left(1, 1, \frac{1}{4} + y^2; 2 \right), \\ F(q^4) &= F(e^{-8\pi y}) = \frac{120(4y)^3}{\pi^2} S(1, 0, 16y^2; 2). \end{aligned} \tag{64}$$

Trivial manipulations suffice to prove

$$S \left(1, 1, \frac{1}{4} + y^2; t \right) = -S(1, 0, y^2; t) + 18S(1, 0, 4y^2; t) - 16S(1, 0, 16y^2; t). \tag{65}$$

Now we prove the formula for $S(1, 0, 36; 2)$ quoted in the introduction (equation (6)). Set $q = -e^{-\pi/3}$ in (43). Using the results above (with $y = \frac{1}{6}$), we conclude

$$\begin{aligned} F(-e^{-\pi/3}) &= \frac{90}{\pi^2} (9S(1, 0, 9; 2) - 8S(1, 0, 36; 2) - 8S(4, 0, 9; 2)) \\ F(e^{-4\pi/3}) &= \frac{2880}{\pi^2} S(4, 0, 9; 2). \end{aligned}$$

We can eliminate $S(4, 0, 9; 2)$ with (63), and $S(1, 0, 9; 2)$ disappears using

$$S(1, 0, 9; t) = (1 + 3^{1-2t}) L_1(t) L_{-4}(t) + L_{-3}(t) L_{12}(t).$$

Putting everything together in (43), and simplifying (a, b, z) with (61), produces the desired formula for $S(1, 0, 36; 2)$.

Next consider (43) when $q = -e^{-\pi/\sqrt{5}}$. Applying (64) and (65) with $y = \frac{1}{\sqrt{20}}$, reduces the formula to a linear combination of $S(1, 0, 20; 2)$, $S(4, 0, 5; 2)$ and $S(1, 0, 5; 2)$.

We can eliminate the latter two sums with

$$\begin{aligned} S(4, 0, 5; t) + S(1, 0, 20; t) &= (1 - 2^{-t} + 2^{1-2t})L_1(t)L_{-20}(t) + (1 + 2^{-t} + 2^{1-2t})L_{-4}(t)L_5(t) \\ S(1, 0, 5; t) &= L_1(t)L_{-20}(t) + L_{-4}(t)L_5(t). \end{aligned}$$

John Zucker provided the first identity, and the second appears in [7]. Thus we arrive at

$$\frac{16\sqrt{5}}{\pi^2}S(1, 0, 20; 2) = \frac{5\sqrt{5}}{3}L_{-20}(2) + \frac{104}{25}L_{-4}(2) - \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a - bn)}{n^3} z^{-n} \quad (66)$$

where

$$\begin{aligned} z &= -8 \left(617 + 276\sqrt{5} + 2\sqrt{5 \left(38078 + 17029\sqrt{5} \right)} \right) \\ a &= \frac{34}{5} + 3\sqrt{5} + \frac{1}{2}\sqrt{\frac{9032}{25} + \frac{808}{\sqrt{5}}} \\ b &= 16 + 7\sqrt{5} + \frac{1}{2}\sqrt{\frac{9728}{5} + \frac{4352}{\sqrt{5}}}. \end{aligned}$$

This formula also converges rapidly, because $z \approx -1.9 \times 10^4$.

We conclude the paper with one final example. To obtain a formula for $S(1, 0, 52; 2)$, set $q = -e^{-\pi/\sqrt{13}}$ in (43). Applying (64) and (65) with $y = \frac{1}{\sqrt{52}}$, reduces the companion series to an expression involving $S(1, 0, 52; 2)$, $S(4, 0, 13; 2)$, and $S(1, 0, 13; 2)$. The latter two sums can be eliminated with

$$\begin{aligned} S(1, 0, 52; t) + S(4, 0, 13; t) &= (1 - 2^{-t} + 2^{1-2t})L_1(t)L_{-52}(t) + (1 + 2^{-t} + 2^{1-2t})L_{-4}(t)L_{13}(t) \\ S(1, 0, 13; t) &= L_1(t)L_{-52}(t) + L_{-4}(t)L_{13}(t). \end{aligned}$$

Zucker provided the first formula, and the second appears in [7]. Therefore, we obtain

$$\frac{16\sqrt{13}}{\pi^2}S(1, 0, 52; 2) = \frac{5\sqrt{13}}{3}L_{-52}(2) + 8L_{-4}(2) - \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a - bn)}{n^3} z^{-n}, \quad (67)$$

where

$$\begin{aligned} z &= -8 \left(3367657 + 934020\sqrt{13} + 90\sqrt{2800274982 + 776656541\sqrt{13}} \right), \\ a &= \frac{4266}{13} + 91\sqrt{13} + \frac{1}{13}\sqrt{2 \left(18194697 + 5046301\sqrt{13} \right)}, \\ b &= 720 + \frac{2595}{\sqrt{13}} + \frac{48}{26}\sqrt{13 \left(23382 + 6485\sqrt{13} \right)}. \end{aligned}$$

Notice that $z \approx -1.07 \times 10^8$, so the formula converges rapidly.

6. CONCLUSION

In conclusion, it might be interesting to try to classify all of the values of $S(A, B, C; 2)$ which can be treated using the ideas in Section 5. It would also be extremely interesting if the methods from Section 3 could be used to say something about 3-dimensional lattice sums such as the Madelung constant.

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REFERENCES

- [1] Digital library of mathematical functions. 2010-05-07. URL <http://dlmf.nist.org/>.
- [2] R. APÉRY, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque*, vol. 61, (1979), 11-13.
- [3] B. C. BERNDT, *Ramanujan's Notebooks, Part III* (Springer-Verlag, New York, 1991).
- [4] B. C. BERNDT, *Ramanujan's Notebooks, Part V* (Springer-Verlag, New York, 1998).
- [5] J. M. BORWEIN and P. B. BORWEIN, *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity*. (New York: Wiley, 1987a.)
- [6] D. B. CHUDNOVSKY and G. V. CHUDNOVSKY, Approximations and Complex Multiplication According to Ramanujan, *Ramanujan Revisited: Proceedings of the Centenary Conference, University of Illinois at Urbana-Champaign, June 1-5, 1987* (Ed. G. E. Andrews, B. C. Berndt, and R. A. Rankin). Boston, MA: Academic Press, pp. 375-472, 1987.
- [7] M. L. GLASSER and I. J. ZUCKER, Lattice Sums, *Perspectives in Theoretical Chemistry: Advances and Perspectives, Vol. 5* (Ed. H. Eyring).
- [8] J. GUILLERA Thesis: Series de Ramanujan (Generalizaciones y conjeturas), Universidad de Zaragoza (Spain) (2007).
- [9] J. GUILLERA, Hypergeometric identities for 10 extended Ramanujan-type series. *Ramanujan J.* **15** (2008), no. 2, 219-234; (arXiv:1104.0396).
- [10] J. GUILLERA, A matrix form of Ramanujan-type series for $1/\pi$. in Gems in Experimental Mathematics T. Amdeberhan, L.A. Medina, and V.H. Moll (eds.), *Contemp. Math.* **517** (2010), Amer. Math. Soc., 189–206; (arXiv:0907.1547).
- [11] J. GUILLERA, WZ-proofs of “divergent” Ramanujan-type series. (arXiv:1012:2681)
- [12] K. HESSAMI PILEHROOD and T. HESSAMI PILEHROOD, Bivariate identities for values of the Hurwitz zeta function and supercongruences. (arXiv:1104.3659).
- [13] S. RAMANUJAN, Modular equations and approximations to π , [Quart. J. Math. **45** (1914), 350-372]. *Collected papers of Srinivasa Ramanujan, 23-29*, AMS Chelsea Publ., Providence, RI, 2000.
- [14] ZHI-WEI SUN, List of conjectural formulas for powers of π and other constants. (arXiv:1102.5649).
- [15] Y. YANG, Apéry limits and special values of L -functions. *J. Math. Anal. Appl.* **343** (2008) 492-513.; (arXiv:0709.1968).
- [16] D. ZAGIER and H. GANGL, Classical and elliptic polylogarithms and special values of L -series. *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, 561615, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.
- [17] D. ZEILBERGER, Closed Form (pun intended!). *Contemp. Math.* **143** (1993), 579-608.
- [18] I. J. ZUCKER and R. C. MCPHEDRAN, Dirichlet L -series with real and complex characters and their application to solving double sums, *Proc. R. Soc. A* **464** (2008), no. 2094, 1405-1422.

- [19] I. J. ZUCKER and M. M. ROBERTSON, Further aspects of the evaluation of $\sum_{(m,n \neq 0,0)} (am^2 + bnm + cn^2)^{-s}$. *Math. Proc. Cambridge Philos. Soc.* **95** (1984), no. 1, 5-13.

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